String Field Theory:
Amplitudes and Tachyon Condensation

Alex Atanasov

Abstract

The purpose of this final paper is to give a review of some of the major developments in open string field theory (OSFT). We begin with a review of Witten's OSFT, and go on to show how this successfully computes the Veneziano amplitude. Then, we demonstrate how using the techniques of string field theory, one can properly show how the bosonic string tachyon condenses.

1 Open String Field Theory

The bosonic string in a flat, 26-dimensional background is described by a sigma model of 26 free scalars X^μ together with a bc ghost system of central charge -26. The action can be written as

$$\frac{1}{2\pi\ell_s^2}\int d^2z\partial X^\mu\bar{\partial}X_\mu + \frac{1}{2\pi}\int d^2z(b\bar{\partial}c + \bar{b}\partial\bar{c}) \tag{1}$$

This system contains, among other things, an anomalously conserved ghost current $-:bc:$ which gives us that

$$\#\text{ zero modes of } c - \#\text{ zero modes of } b = 3 - 3g.$$

The bc system has two degenerate ground states $|\downarrow\rangle, |\uparrow\rangle$ which correspond to modes $c_1|1\rangle$ and $c_0c_1|1\rangle$.

The full system has a fermionic supersymmetry with a conserved current

$$j_B = c(T^X + \frac{1}{2}T^{gh}) + c\partial^3 c \tag{2}$$

that is a primary operator exactly when the spacetime dimension is 26. Its corresponding conserved charge $Q_B = \int \frac{d^2z}{2\pi i}j_B$ is a nilpotent operator on the Hilbert space $Q_B^2 = 0$. Looking at all closed states modulo exact (i.e. null) states, we obtain the *BRST cohomology* on the worldsheet.

Physical states–corresponding to the insertion of vertex operators–are then represented by BRST cohomology classes subject to the additional *Siegel gauge* constraint $b_0|\psi\rangle = 0$. That is, they are states on the cylinder with ghost number $-1/2$. Equivalently, we will view these states on the upper half plane as ghost number 1, which is more standard in the modern literature. From now on we will use the upper half plane ghost number.

The connected component of the S-matrix can then be written as a sum over open-string world sheet topologies

$$\sum_{g,b}\mathcal{A}_{g,b}[V_1(k_1)\ldots V_n(k_n)]$$

where each $A_{g,b}$ is understood as an n-point conformal correlator of vertex operators. in this essay we will be focusing on the tree-level open string amplitudes. $g = 0, b = 1$.

In going to string field theory, our variable will the the string field operator Ψ which creates a state of ghost number 1 that need *not* be BRST closed. In the the Schrodinger picture of open string field theory we can imagine cutting out a small half-disk around $z = 0$ where $\Psi(0)$ is inserted. This is then a functional $\Psi[X, b, c]$ on the matter and ghost fields on the WS. Just as in QFT, these first-quantized wave functionals are promoted to dynamical fields.

$Q_B\Psi = 0$ should then emerge as an equation of motion from a more general *string field action.* This leads us to expect a leading-order string field action of the form:

$$S(\Psi) = \frac{1}{2}\langle\langle\Psi|Q_B|\Psi\rangle$$

here $\langle\langle\Psi|$ denotes the BPZ conjugate of $|\Psi\rangle$.

If $|\Psi\rangle$ is not an on-shell state the n-point string field correlation functions will in general be dependent on the choice of conformal frame used. In other words, we need the explicit data of our local holomorphic coordinate at each point of insertion of our manifold[1]. This space of conformal frames gives a fiber bundle $\mathcal{P}_{g,b,n} \to \mathcal{M}_{g,b,n}$ over moduli space. We must then choose a section of this bundle for each g, b, n. Additionally this choice of sections must be compatible upon factorization.

Different ways of introducing coordinates correspond to field redefinitions in the action of the string field theory. The physical observables, of course, do not depend on these choices. In the next section, we will choose a coordinate system for the 3-point vertex which gives rise to a simple cubic action. This is Witten's *open string field theory*, introduced in [1].

2 The Three Point Vertex

Important in the construction of the 3-point vertex is that there is a copy of $\mathbb{Z}_3 \subset \mathrm{PSL}_2(\mathbb{R})$. One such copy is given by the fractional linear transformation $F(z) = \frac{1}{1-z}$. It is easy to see that $F(F(F(z))) = z$, with a fixed point corresponding to $(-1)^{1/3} = \frac{1}{2}(1 + i\sqrt{3})$. We could have picked a different representative in $\mathbb{H}$, giving a different map F that is still order three. Using this, we can define the three point vertex. Our mapping will be as in the figure below:

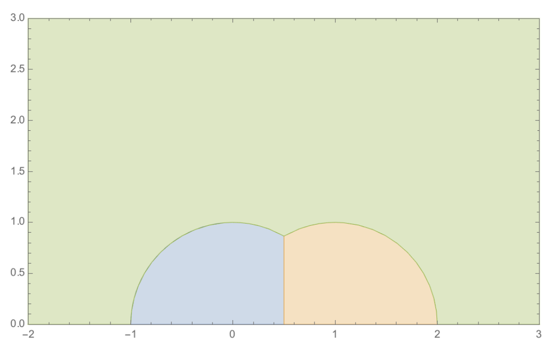


Figure 1: The upper half disk is mapped under h_1, h_2, h_3 to the blue, yellow, and green regions illustrated in this figure. $z = 0$ is mapped then to $0, 1\infty$ in the above region, and i is mapped to the mutual interaction point $(-1)^{1/3}$.

Mapping the upper half-disk to the first (blue) region is given by the simple transformation:

$$h(z) = e^{-i\pi/3} \frac{(\frac{1+iz}{1-iz})^{2/3} - 1}{(\frac{1+iz}{1-iz})^{2/3} + e^{i\pi/3}} \tag{3}$$

Then we can obtain the map from the half-disk to the other three regions by combining this with $F(z)$. This gives $h_1 = h, h_2 = F \circ h_1, h_3 = F \circ h_2$

The insertions of a state at $z = 0$ on the disk will correspond to insertions of 3 states at $w = 0, 1, \infty$ on the corresponding space that the disk is mapped to. Similarly, $z = i$ will be mapped to the mutual interaction point $(-1)^{1/3}$. We take the metric on the half-disk to be the pullback of the flat metric on three strips coming together with a curvature singularity at their intersection point, as illustrated below (image taken from Rastelli's contribution to [2]):

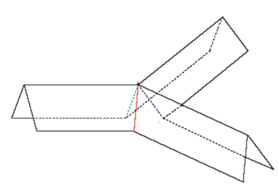


Figure 2: The three point vertex in the cubic OSFT. Note that this surface is everywhere flat except for a curvature singularity at the intersection of the three sheets.

[1]I'm implicitly using Schiffer variation here, so that only reparameterization of coordinates on the disks around the points of insertion matters for distinguishing different coordinate systems.

Using this map, we can now explicitly write the OSFT action as

$$S = -\frac{1}{g_o^2}\left(\frac{1}{2}\langle I \circ \Psi(0) Q_B \Psi(0)\rangle + \frac{1}{3}\langle h_1 \circ \Psi(0) h_2 \circ \Psi(0) h_3 \circ \Psi(0)\rangle\right) \tag{4}$$

here $h_i \circ \Psi$ denotes the conformal transformation of Ψ under h_i and $I(z) = -1/z$. If Ψ is a primary of weight Δ this transforms as $(h_i'(0))^\Delta \Psi(h_i(0))$. For a non-primary field, this is a bit more complicated.

Note that on the disk, *both* of the terms in the action have ghost number 3, as required to cancel the anomaly in the ghost current. For this reason, it might be reasonable to expect that this action alone is already sufficient for generating string interactions. We could not, for example, add a term like $\langle \Psi(0)\Psi(0)\Psi(0)\Psi(0)\rangle_{D_2}$ to this action, as that would violate ghost number on the disk.

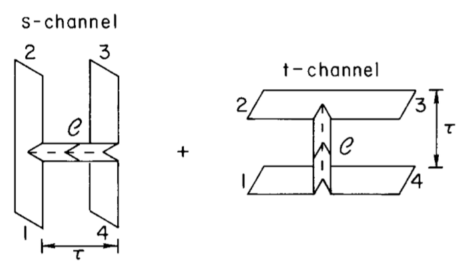
3 The Veneziano Amplitude

Given the 3-point interaction vertex, one can apply a formalism similar to that in QFT to arrive at higher point amplitudes. It is instructive to compute the 4-point Veneziano amplitude, and appreciate how the nontrivial structure of this amplitude, which in QFT requires summing diagrams from the contributions of an infinite number of high-spin particles, comes from a simple s+t-channel calculation in string field theory.

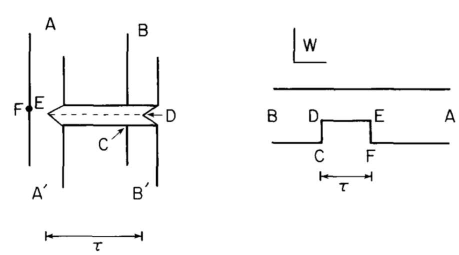
We follow Giddings' 86 paper [3] in this section, and take all figures from therein unless explicitly said otherwise. We have also used [4] to confirm various subtle aspects of the calculation.

As in the last section, we consider only one cubic interaction vertex in our string field theory. Graphically, we have that Feynman diagrams are obtained from gluing together equal width strips with strip "propagators" of variable internal length, namely the Schwinger parameters τ. The curvature singularities are the interaction points only (c.f. Figure 2), which requires a ghost insertion to cancel the anomaly in the currrent.

We now turn to the Veneziano amplitude. In string field theory it will be the sum of an s and t channel diagram as below.



This can be conformally mapped into the following picture, consisting of two copies of this w plane, connected by a branch cut along CD and EF.



The mapping taking C, D, E, F, A, B on one sheet to $\infty, -\delta^2, -\gamma^2, 0, \alpha^2, \beta^2$ (resp.) on the upper half plane is nothing more than a standard Christoffel transformation:

$$\frac{dw}{dt} = \frac{1}{2} N \frac{\sqrt{t+\gamma^2}\sqrt{t+\delta^2}}{\sqrt{t}(t-\alpha^2)(t-\beta^2)}$$

To get the second sheet, we take $z^2 = t$ giving a final mapping:

$$\frac{dw}{dz} = N \frac{\sqrt{z^2+\gamma^2}\sqrt{z^2+\delta^2}}{(z^2-\alpha^2)(z^2-\beta^2)} \tag{5}$$

This gives us our desired relationship between w and z. let us impose some constraints from the physics. In the reparameterization gauge that we have chosen, the strip width of each incoming and outgoing string is π, so that a half-contour integral over any one of the insertions at $\pm\alpha, \pm\beta$ should give πi. Consequently we get $\alpha\beta = \gamma\delta$. We further have the freedom to rescale the z coordinate and remain in $\mathbb{H}$, so we can change α, β so that $\alpha\beta = \gamma\delta = 1$. This leaves two free parameters: α, γ. At this stage we see that the invariant cross-ratio for the vertex operators is

$$x = \left(\frac{1-\alpha^2}{1+\alpha^2}\right)^2 \tag{6}$$

The last two conditions are a little trickier:

- The strip FE has length equal to half of A's, namely $\pi/2$
- The strip DE has length equal to the Schwinger parameter τ

Enforcing the first constraint requires explicit integration along the $(i\gamma, i\delta)$ strip. Taking $\sin^2\theta_1 = \frac{\alpha^{-2}}{\alpha^{-2}+\gamma^2}$ and $\sin^2\theta_2 = \frac{\alpha^2}{\alpha^2+\gamma^2}, k = \gamma^2, k' = \sqrt{1-\gamma^4}$ we can write these two constraints as:

$$\begin{aligned} \frac{1}{2} &= \Lambda_0(\theta_1, k) - \Lambda_0(\theta_2, k) \\ \frac{\tau}{2} &= K(k')(Z(\theta_2, k') - Z(\theta_1, k')) \end{aligned} \tag{7}$$

Here Λ_0, Z are Heumann's Lambda and Jacobi's Zeta function respectively. These are known functions expressible in terms of elliptic integrals. All these constraints together allow us to explicitly write $\alpha, \beta, \gamma, \delta$ in terms of τ.

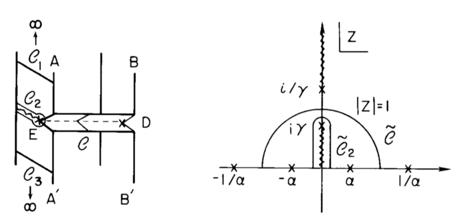
The interesting physics comes from looking at the limiting cases. Consider now the limit $\tau \to \infty$. This requires $K(\sqrt{1-\gamma^4}) \to \infty \Rightarrow \gamma \to 0$. Then $\Lambda_0(\theta_i, 0) = \sin(\theta_i)$ implies that we must have $\alpha \to 0$, and consequently $x \to 1$.

In the other limit $\tau \to 0$, assuming $\theta_2 \neq \theta_1$, we similarly get $\gamma \to 1$. This will equivalently require $\theta_1 - \theta_2 = \pi/4$, which algebraically implies $x = 1/2$. Thus, we get that the s channel contribution reproduces exactly *half* the integration region $1/2 < x < 1$. The t-channel will give us exactly the other half. But are we really integrating the right thing to give us Veneziano?

The answer is yes. For one, taking the Jacobian to express $d\tau$ in terms of $d\alpha$ gives (after messy algebra):

$$d\tau = -\frac{2\pi}{K(\gamma^2)} \frac{d\alpha}{\sqrt{1+\alpha^2\gamma^2}\sqrt{\alpha^2+\gamma^2}}$$

Further, the integral with ghost insertions (4 c-ghosts 1 b-ghost) in terms of bosonized fields $c = e^{i\phi(z)}, b = e^{-i\phi(z)}$ looks like



$$\int \frac{dz}{2\pi i}\frac{dz}{dw} e^{i\sum_a J^a\phi_a} e^{-i\phi(z)} = \frac{K(\gamma^2)}{2\pi}\sqrt{1+\alpha^2\gamma^2}\sqrt{\alpha^2+\gamma^2}\frac{1-\alpha^4}{\alpha^3} \tag{8}$$

Here J^a is $+1$ on each of the four vertex insertions on the boundary, corresponding to the c ghost. A lot of remarkable cancelation happens (this cancelation between the Jacobian and the ghost integral is generic) and we are left with:

$$-2\int d\alpha \frac{1-\alpha^4}{\alpha^3}\exp\left(-\sum_{i<j} p_j p_k \langle X(z_j)X(z_k)\rangle\right) = -\frac{1}{4}\int_{1/2}^{1} dx\, x^{2p_1\cdot p_2}(1-x)^{2p_2\cdot p_3} \tag{9}$$

This is exactly half of the Veneziano amplitude. Adding the t channel gives the other half.

From this, we see that the s and t channel string field diagram perfectly cover the region in Veneziano amplitude where $0 < x_4 < 1$. This is part of a more general story. The open string Feynman diagrams provide a one-fold cover for the moduli space of *open string* diagrams. Zweibach showed this explicitly in [5]. He did this by considering for a given worldsheet Σ a Hermitian metric that minimizes the area of Σ subject to the constraint that *all nontrivial Jordan curves have length* $\geqslant \pi$. Here a nontrivial Jordan curve is one that cannot be contracted while keeping its endpoints fixed on a given boundary.

Indeed, these worldsheets resulting from this minimization problem will have a geometry of being everywhere flat except at singular points, exactly mirroring the 3-vertex worldsheet constructions in Witten's OSFT. Thus, for a given Riemann surface of fixed moduli, SFT associates one and only one string diagram to it. This diagram comes with a metric of *minimal area* subject to suitable length conditions. This section of $P_{g,b,n}$ inherent in Witten's OSFT is what yields the particularly simple form of the action that we have seen.

A good way to highlight the conceptual difference and difficulties between closed and open string field theories is by contrasting the straightforward decomposition of the Veneziano amplitude with a similar calculation on the sphere: the Virasoro-Shapiro amplitude. For the disk amplitude, the relevant moduli space that we integrate over is the real line corresponding to the fourth vertex insertion. We see that it is possible to choose consistent coordinates around each of the first three punctures to provide a one-fold cover of that real line. Consequently, the full amplitude can be expressed just in terms of the tree-level 3-point interactions. This is not the case for the Virasoro-Shapiro amplitude, where the open disks around each of the first three insertions cover only a portion of the full sphere, and consequently we must add higher order terms to the string field action to recover the full amplitude. This speaks to the relative simplicity of open string field theory.

4 Tachyon Condensation in String Field Theory

In [6], Sen showed that the tachyon potential for a D-brane in the bosonic string theory takes a universal form as a function of the tachyon field

$$V(T) = Mf(T)$$

Here M is the mass of the D-brane and $f(T)$ is *universal* in the sense that it does not depend on the background that the D-brane is in, or even the dimension of the D-brane. We will not go over the arguments for universality in this paper. Rather, we will show in the framework of string field theory that there exists a state $|T_c\rangle$ where $f(T)$ achieves a stationary point, and that in fact $f(T_c) = -1$. This will imply that, upon reaching T_c, we will arrive at $V(T) = -M$. The fact that the potential is exactly opposite to the mass of brane implies that the D-brane has completely condensed to what Sen calls the *non-perturbative vacuum* of the bosonic string.

Following equation (4), $f(T)$ is given in terms of the string field as

$$f(T) = 2\pi^2\left(\frac{1}{2}\langle I\circ\mathcal{T}(0)Q_B\mathcal{T}(0)\rangle + \frac{1}{3}\langle h_1\circ\mathcal{T}(0)h_2\circ\mathcal{T}(0)h_3\circ\Psi(0)\rangle\right) \tag{10}$$

The normalization is chosen so that, given $M = \frac{1}{2\pi^2 g_o^2}$ for the D-brane mass, we recover the potential through $Mf(T)$. Here $\mathcal{T}$ is a two-dimensional field that creates the string-field state $|T\rangle$ out of the $\mathrm{SL}_2(\mathbb{R})$-invariant vacuum $|1\rangle$, ie $\mathcal{T}\,|1\rangle = |T\rangle$.

We proceed via Sen's formalism of level truncation, and follow the paper [7]. Let $\mathcal{H}$ be the Hilbert space of states (not just physical) of ghost number 1 in the 2D CFT of the open string (ie 26 matter fields X^μ and the bc system). Consider the subspace $\mathcal{H}_1$ given by acting on the $\mathrm{SL}_2(\mathbb{R})$ vacuum by the b, c modes as well as the modes L_n of the matter theory. That is

$$\mathcal{H}_1 := \mathrm{Span}\{c_{-i_1}\dots c_{-i_p}b_{-j_1}\dots b_{-j_q}L_{-k_1}\dots L_{-k_r}\,|1\rangle\,|i_l \geqslant -1, j_l, k_l \geqslant 2\}$$

The subspace $\mathcal{H}_1$ is background independent, and contains it it the zero-momentum tachyon state. And take $\mathcal{H}_2 = \mathcal{H} - \mathcal{H}_1$ [2]. Importantly, we have that conformal transformations do not mix $\mathcal{H}_1$ and $\mathcal{H}_2$, nor does the kinetic term Q_B have matrix nonzero elements between $\mathcal{H}_1$ and $\mathcal{H}_2$. Finally, the variation of S along $\mathcal{H}_2$ for a state $|\Psi\rangle \in \mathcal{H}_1$ is always zero, so that states in $\mathcal{H}_2$ appear quadratic or higher-order in the action.

Thus we can consistently truncate $\mathcal{H} \to \mathcal{H}_1$. All fields in $\mathcal{H}_1$ can take expectation values. Upon taking the normalization convention $\langle\langle 1|\, c_{-1}c_0c_1\,|1\rangle = 1$, Sen's conjecture takes the form $f(T_c) = -1$ with $f(0) = 0$.

The lowest lying state in $\mathcal{H}_1$ is $tc_1\,|1\rangle = tc(0)\,|1\rangle$. Henceforth, the "level" of a state will always be measured relative to this state.

[2] The notation "$-$" that Sen uses is ambiguous here. I believe it suffices for our purposes to define this as an orthogonal complement with respect to the inner product, even if it is indefinite.

4.1 Level 0

For the zeroth approximation, we take $tc_1 |1\rangle = tc(0) |1\rangle$ and calculate the quadratic and cubic terms.

$$\begin{aligned}\langle I \circ \mathcal{T}(0) Q_B \mathcal{T}_0 \rangle &= t^2 \langle\langle 1| c_{-1}(-c_0) c_1 |1\rangle = -t^2 \\ \langle h_3 \circ \mathcal{T}(0) h_2 \circ \mathcal{T}(0) h_1 \circ \mathcal{T}(0) \rangle &= t^3 \langle\langle 1| \frac{1}{r} c_{-1} \frac{1}{r}(-c_0) \frac{1}{r} c_1 |1\rangle = -\frac{t^3}{h_1'(0)\, h_2'(0)\, (1/h_3)'(0)}\end{aligned} \tag{11}$$

Take $r^3 = -h_1'(0)(h_2-1)'(0)(1/h_3)'(0) = \left(\frac{4}{3\sqrt{3}}\right)^3$. From this zeroth order truncation, we get a tachyon potential:

$$f(T) = 2\pi^2 \left(-\frac{1}{2} t^2 + \frac{1}{3} \frac{t^3}{r^3} \right) \tag{12}$$

This has its local minimum at $t_c = r^3$ giving $2\pi^2(-\frac{1}{2}r^2 + \frac{1}{3}) = -\frac{2^{12}}{3^{10}}\pi^2 \approx -0.684$. This is already almost 70% to the goal of -1. Sen remarks the the cubic open string field theory performs the best among all choices of OSFT at zeroth order. This is because of the fact that the parameter r, related to the mapping radius of the disks defining the three string vertex, is maximal for the coordinate system of the cubic string field theory, and consequently $|f(t_c)|$ is maximal.

4.2 Higher Levels and Twist Symmetry

The higher levels are only technically, but not conceptually, more difficult. We will illustrate the next level so as to give a more complete picture.

Before going forward, an important step that simplifies the higher-level calculations is to note that the action (4) has a $\mathbb{Z}_2$ symmetry

$$|\Psi\rangle \to (-1)^{L_0+1} |\Psi\rangle$$

which leaves even level states invariant. To see this, note the following relationship with $\tilde{I}(y) = 1/y$:

$$h_1(-z) = \tilde{I} \circ h_3, \quad h_2(-z) = \tilde{I} \circ h_2, \quad h_3(-z) = I \circ h_1$$

Then, with $M(z) = -z$, using the fact that the correlation functions are invariant under $\mathrm{SL}_2(\mathbb{R})$ and worldsheet parity, we have[3]

$$\begin{aligned}\langle h_1 \circ \mathcal{T}(0)\, h_2 \circ \mathcal{T}(0)\, h_3 \circ \mathcal{T}(0) \rangle &= \langle (-1)^{L_0} h_1 \circ M \circ \mathcal{T}(0)\, h_2 \circ M \circ \mathcal{T}(0)\, h_3 \circ M \circ \mathcal{T}(0) \rangle \\ &= \langle (-1)^{L_0} \tilde{I} \circ h_3 \circ \mathcal{T}(0)\, \tilde{I} \circ h_2 \circ \mathcal{T}(0)\, \tilde{I} \circ h_1 \circ \mathcal{T}(0) \rangle \\ &= \langle (-1)^{L_0+1} h_1 \circ \mathcal{T}(0)\, h_2 \circ \mathcal{T}(0)\, h_3 \circ \mathcal{T}(0) \rangle\end{aligned}$$

Odd levels must thus enter the action in pairs, and setting these fields to zero satisfies the equation of motion. We can thus consistently truncate to even levels.

At level 2 the states that contribute are

$$c_{-1} |1\rangle, \quad L_{-2} c_1 |1\rangle, \quad b_{-2} c_0 c_1 |1\rangle.$$

By imposing Seigel gauge $b_0 |\Psi\rangle = 0$, we can simplify this further. It is not clear that this gauge choice can be justified in the interacting theory. First, note that in the linearized theory for a state at level $2n$, defining

$$|\tilde{T}^{(2n)}\rangle = |T^{(2n)}\rangle - \frac{1}{2n-1} Q_B^{-1} b_0 |T^{(2n)}\rangle$$

gives a state satisfying $b_0 |\tilde{T}^{(2n)}\rangle = 0$. Thus, we can always pick a cohomology class that satisfies Seigel gauge. In the linearized theory, we are free to do this level-by-level. Moreover, this gauge is sufficient in that it leaves no residual gauge direction. If there were such an additional direction satisfying Seigel gauge $|\eta^{2n}\rangle$, then $b_0 |\eta^{2n}\rangle$ together with $Q_B |\eta^{2n}\rangle$ gives $L_0 |\eta^{2n}\rangle = 0$, but since $|\eta^{2n}\rangle$ is at level $2n$ we must have $L_0 |\eta^{2n}\rangle = 2n-1$. Under the assumption that the interactions are small enough, we can continue to keep this gauge in the interacting theory.

[3]Thanks to Atakan for pointing out this proof.

Thus, at level 2 we can write an arbitrary state in Seigel gauge as

$$
\begin{aligned}
|T\rangle &= tc_1 |1\rangle + uc_{-1} |1\rangle + \frac{v}{\sqrt{13}} L_{-2} c_1 |1\rangle \\
&= tc(0) |1\rangle + u \frac{1}{2} \partial^2 c |1\rangle + \frac{v}{\sqrt{13}} cT(0) |1\rangle
\end{aligned}
\tag{13}
$$

where the coefficient $v/\sqrt{13}$ was chosen as in Sen's paper, for convenience. Before plugging this into the potential, let us only keep terms in $f(T)$ up to a certain level. Note that the quadratic term will necessarily involve terms of level 4 at this truncation of $|T\rangle$, so it makes sense to truncate the cubic term also to level 4. It will thus involve terms of the form $t^3, t^2u, t^2v, tu^2, tv^2$ and tuv. In general when truncating the state at level n we can define such a level $2n$ approximation to the potential f^{2n}.

Given that the Schwarzian vanishes for inversion I, we get that the BPZ conjugate of $|T\rangle$ is

$$
\begin{aligned}
\langle\langle T| &= \langle\langle 1| tc_{-1} + \langle\langle 1| uc_1 + \langle\langle 1| \frac{v}{\sqrt{13}} L_2 c_{-1} \\
\Rightarrow \langle\langle TQ_bT\rangle &= t^2 \langle\langle 1| c_{-1} L_0 c_0 c_1 |1\rangle + u^2 \langle\langle 1| c_1 L_0 c_0 c_{-1} |1\rangle + \frac{v^2}{13} \langle\langle 1| L_2 c_{-1} L_0 c_0 L_{-2} c_1 |1\rangle \\
&= -t^2 - u^2 + v^2
\end{aligned}
\tag{14}
$$

The cubic term is involved, but can be done. The main complication comes from the fact the the fields are no longer primaries. We will have

$$
h \circ \frac{1}{2}\partial^2 c(0) = \frac{1}{2}\partial^2 \left(\frac{1}{h'(0)} c(f(0)) \right) \qquad h \circ cT(0) = h'(0) cT(f(0)) + \frac{1}{h'(0)} S(f,0) c(f(0))
\tag{15}
$$

where S is the Schwarzain derivative $-\frac{26}{12}\left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right)$. At level 2 with interaction up to level 4, we get

$$
f(T) = 2\pi^2 \left(-\frac{1}{2}t^2 - \frac{1}{2}u^2 + \frac{1}{2}v^2 + \frac{3^3\sqrt{3}}{2^6} t^3 + \frac{11 \times 3\sqrt{3}}{2^6} t^2 u - \frac{5 \times 3\sqrt{39}}{2^6} t^2 v + \frac{19}{2^6\sqrt{3}} tu^2 + \frac{7 \times 83}{2^6 3\sqrt{3}} tv^2 - \frac{11 \times 5\sqrt{13}}{2^5 \times 3\sqrt{3}} tuv \right)
\tag{16}
$$

This indeed has a critical point T_c given by $t \approx 0.524, u \approx 0.172, v \approx 0.187$ and $f(T_c) \approx -0.949$. This brings us up to 95%.

For calculating the interaction at level 6, we do not need to add any new terms to $|T\rangle$, since odd terms are excluded. We get additional u^3, v^3, uv^2, u^2v contributions to the cubic interaction term, and find $f(T_c) \approx -0.959$, a marginal improvement. Finally Sen calculates the potential with the state $|T\rangle$ truncated at level 4, keeping up to level 8 interaction terms. This gives a T_c so that $f(T_c) = -0.9864$, 99% of the way there. It does indeed appear that this sequence converges quite rapidly to the desired value of -1.

4.3 Subsequent Progress

We have seen strong numerical evidence that the tachyon potential exactly cancels the D-brane tension, and surprisingly we only needed to use low-lying levels in the framework in string field theory to see this highly nonperturbative effect. Moeller and Taylor [8] later did this to level 10 truncation with level 20 interaction, giving $f(T_c) = -0.99912$. Finally, Schnabl [9] gave an analytic solution, proving Sen's conjecture.

References

[1] Edward Witten. Non-commutative geometry and string field theory. *Nuclear Physics B*, 268(2):253–294, 1986.

[2] Jean-Pierre Françoise, Gregory L Naber, and Sheung Tsun Tsou. *Encyclopedia of mathematical physics*, volume 2. Elsevier Amsterdam etc., 2006.

[3] Steven B Giddings. The veneziano amplitude from interacting string field theory. *Nuclear Physics B*, 278(2):242–255, 1986.

[4] Michio Kaku. *Strings, conformal fields, and M-theory.* Springer Science & Business Media, 2012.

[5] Barton Zwiebach. A proof that witten's open string theory gives a single cover of moduli space. *Communications in mathematical physics*, 142(1):193–216, 1991.

[6] Ashoke Sen. Universality of the tachyon potential. *Journal of High Energy Physics*, 1999(12):027, 2000.

[7] Ashoke Sen and Barton Zwiebach. Tachyon condensation in string field theory. *Journal of High Energy Physics*, 2000(03):002, 2000.

[8] Nicolas Moeller and Washington Taylor. Level truncation and the tachyon in open bosonic string field theory. *Nuclear Physics B*, 583(1-2):105–144, 2000.

[9] Martin Schnabl et al. Analytic solution for tachyon condensation in open string field theory. *Advances in Theoretical and Mathematical Physics*, 10(4):433–501, 2006.